REMARK. Mr. Drummond raises the question as to whether Mr. J. F. Travis's solution of problem 87, Arithmetic, is strictly an arithmetical solution. To my mind it is strictly an algebraical solution. A pure arithmetical solution of a problem would involve only the operations of addition, subtraction, multiplication, division, involution, and evolution, without the use of equations. A solution in which the result sought is represented by some character, and then this character operated upon until certain conditions of the problem are fulfilled, which conditions are then stated in the form of an equation from which the numerical value of the character is to be determined, is an algebraic solution. It is immaterial what sort of a character is used, whether it be $(\frac{2}{2})$, $\frac{3}{3}$, x, ϕ, or any other character. However, the solution referred to is a very good one, and by the use of such solutions students in arithmetic are given, unconsciously to themselves, a most excellent preparation for the study of algebra. The mathematician is often called upon to solve problems in a certain way. When a problem is proposed and the restriction put upon it, viz., that it be solved by arithmetic, or algebra, or geometry, the problem often becomes impossible. From such unfortunate restrictions, has arisen the idea of the insolvability of the three famous problems of geometry, viz., the Trisection of an Angle, the Duplication of the Cube, and the Quadrature of the Circle. These problems are each easily solved if the solutions are not restricted to the use of the straight edge and compass only. But with these restrictions they are absolutely unsolvable.

There are many problems whose solutions cannot be effected when restricted in the way previously mentioned, but those referred to above are the only ones that have become famous.

ALGEBRA.

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81. II. Solution by C. W. M. BLACK, A. M., Professor of Mathematics, Wesleyan Academy, Wilbraham, Mass.

[See problem and solution I, in April number, page 105.] The proposition cannot be proved unless r is integral and positive, as can be shown by substitution of numerical values.

Consider the only two fractions in whose denominators any factor as (a_1-a_2) appears, putting them in the form

$$(a_1{}^r)/[(a_1-a_2)(a_1-a_3)(a_1-a_4)\ \ldots\ldots(a_1-a_n)]-(a_2{}^r)$$
$$/[(a_1-a_2)(a_2-a_3)(a_2-a_4)\ \ldots\ldots(a_2-a_n)]=(a_1{}^r)$$
$$/[(a_1-a_2)(a_1{}^{n-2}P_1a_1{}^{n-3}+P_2a_1{}^{n-4}-\ldots\ldots\pm P_{n-2})]$$
$$-(a_2{}^r)/[(a_1-a_2)(a_2{}^{n-2}P_1a_2{}^{n-3}+P_2a^{n-4}-\ \ldots\ldots\pm P_{n-2})],$$

where P_k=the sum of the products of a_3, a_4, $\ldots\ldots a_n$ taken k at a time.

Combining, we have

$$[a_1{}^r(a_2{}^{n-2}-P_1a_2{}^{n-3}+P_2a_2{}^{n-4}-\ldots\ldots\pm P_{n-2})$$
$$-a_2{}^r(a_1{}^{n-2}-P_1a_1{}^{n-3}+P_2a_1{}^{n-4}-\ldots\ldots P_{n-2})]$$
$$/[(a_1-a_2)(a_1{}^{n-2}-P_1a^{n-3}+P_2a_1{}^{n-4}-\ldots\ldots\pm P_{n-2})$$
$$(a_2{}^{n-2}-P_1a_2{}^{n-3}+P_2a_2{}^{n-4}-\ldots\ldots\pm P_{n-2})]\ \ldots\ldots\ldots.(1).$$

Put the numerator of (1) in the form

$$(a_1{}^ra_2{}^{n-2}-a^ra_1{}^{n-2})-P_1(a_1{}^ra_2{}^{n-3}-a_2{}^ra_1{}^{n-3})+\ldots\ldots\ \pm P_{n-2}(a_1{}^r-a_2{}^r)$$
$$=a_1{}^{n-2}a_2{}^{n-2}(a_1{}^{r-n+2}-a_2{}^{r-n+2})-P_1a_1{}^{n-3}a_2{}^{n-3}(a_1{}^{r-n+3}-a_2{}^{r-n+3})$$
$$+\ldots\ldots\pm P_{n-2}(a_1{}^r-a_2{}^r)\ldots\ldots(2).$$

If n is not greater than $r+2$ each group of (2) and consequently the whole expression is divisible by (a_1-a_2). If $n>r+2$, let $n=r+s$; then change (2) to the form

$$a_1{}^ra_2{}^r[(a_2{}^{n-2-r}-a_1{}^{n-2-r})-P_1(a_2{}^{n-3-r}-a_1{}^{n-3-r})+\ldots\ldots$$
$$\pm P_{s-2}(a_2{}^{n-s-r}-a_1{}^{n-s-r})]\mp[P_{s-1}a_1{}^{n-s-1}a_2{}^{n-s-1}(a_1{}^{r-n+s+1}-a_2{}^{r-n+s+1})$$
$$-P_sa_1{}^{n-s-2}a_2{}^{n-s-2}(a_1{}^{r-n+s+2}-a_2{}^{r-n+s+2})$$
$$+\ldots\ldots\pm P_{n-2}(a_1{}^r-a_2{}^r)]\ldots\ldots(3),$$

the term in the second group of (3) being the same as the corresponding term of (2). Each group in (3) is also divisible by (a_1-a_2). Accordingly in all cases (1) can be reduced to a form in which (a_1-a_2) is not a factor of the denominator, and as the two fractions forming (1) are the only ones that contain (a_1-a_2) in their denominators, the original expression need not contain (a_1-a_2) in its denominator; that is, (a_1-a_2) will divide into the numerator formed by adding the fractions as they stand.

In like manner we prove that any other factor (a_2-a_3), etc., will divide into the numerator, or the numerator will be divisible by the entire lowest common denominator.

Now if $r<n-1$ each fraction, and consequently the sum of all, will have a numerator of lower degree in a_1, a_2, a_3, etc., than the denominator. But as the numerator is divisible by the denominator, this is possible only when the numerator equals zero.

If $r=n-1$, numerator and denominator will have same degree, and the quotient can be only a numerical factor. Now in the numerator $a_1{}^r$ has for its coefficient the product of all factors not containing a_1, which same coefficient it has in the expansion of the denominator. Therefore the quotient must equal 1.

If $r=n$ the numerator is of a degree 1 higher than the denominator and the quotient must be of the first degree. In the numerator the coefficient of $a_1{}^r$ is the same as the coefficient of $a_1{}^{n-1}(=a_1{}^{r-1})$ in denominator, these being the highest powers of a_1 in each. Then one term of the quotient must be a_1. In

like manner we show that a_2, a_3, ...a_n must all be true of quotient, and as the expression is symmetrical with respect to these, the value must be

$$a_1+a_2+a_3+\dots\dots+a_n.$$

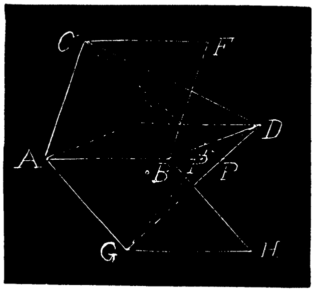
GEOMETRY.

87. **Proposed by WALTER HUGH DRANE, A. M., Professor of Mathematics, Jefferson Military Academy, Washington, Miss.**

Given any two straight lines in space, AB, CD, which do not intersect. So construct upon one of the lines as base, a triangle, having its vertex in the other line, such that its perimeter shall be a minimum.

I. **Solution by the PROPOSER.**

Let AB and CD be the given straight lines. Pass planes through the line AB and the points C and D. In the plane of $ABDE$, inclined the same way and making the same angle with the line AB as $ABFC$, construct a parallelogram equal to $CABF$. Draw DG intersecting AB produced in P. Join PC. Then PCD is the required triangle.



Proof. Take any other point P' in the line AB. Join $P'D$, $P'C$, and $P'G$. Triangle $P'CA$=triangle $P'GA$ and triangle PCA=triangle PGA. Two sides and included angle being equal in each case. $\therefore$ $P'C=PG$ and $PC=PG$.

Now $P'D+P'G>PD+PG$.

$\therefore$ $P'D+P'C>PD+PC$. Q. E. D.

By passing planes through CD and the points A and B, by a similar construction we may construct a minimum-perimeter triangle upon AB as base with its vertex in CD.

Also solved by *F. R. HONEY.*

II. **Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.**

Let $x+b(a-x)=cy$ be the equation to EF with AB and AY as axes, and AB, EF the given lines, $AB=a$. Then if $\cot\theta+b\cot\varphi=c$ the vertex C will move on EF. Let $AC=r$, $CB=s$.

$$\text{Then } r+s=\frac{a(\sin\theta+\sin\varphi)}{\sin(\theta+\varphi)}=\text{minimum}\dots\dots\dots\dots\dots\dots\dots\dots (1).$$

$$\cot\theta+b\cot\varphi=c\dots\dots\dots\dots\dots\dots\dots\dots(2).$$

$$\text{From (1), } \frac{d\theta}{d\varphi}=-\frac{\sin\theta}{\sin\varphi}, \text{ from (2), } \frac{d\theta}{d\varphi}=-\frac{b\sin^2\theta}{\sin^2\varphi}.$$

$\therefore$ $\sin\varphi=b\sin\theta$, this in (2) gives